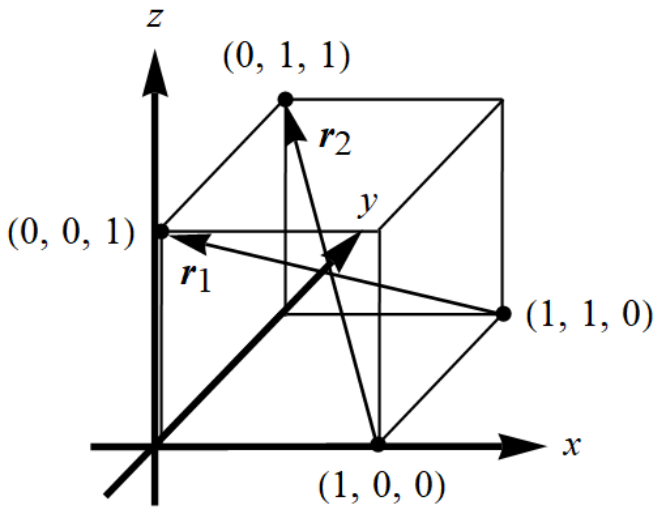
Problem 1.3

Find the angle between the body diagonals of a cube.

Solution

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be displacement vectors for body diagonals that have their starting points on adjacent vertices in a unit cube, for example, those shown below.



$$\mathbf{r}_1 = \langle 0,0,1\rangle - \langle 1,1,0\rangle = \langle -1,-1,1\rangle$$
$$\mathbf{r}_2 = \langle 0,1,1\rangle - \langle 1,0,0\rangle = \langle -1,1,1\rangle$$

Use the definition of the dot product between $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1||\mathbf{r}_2| \cos\theta$$

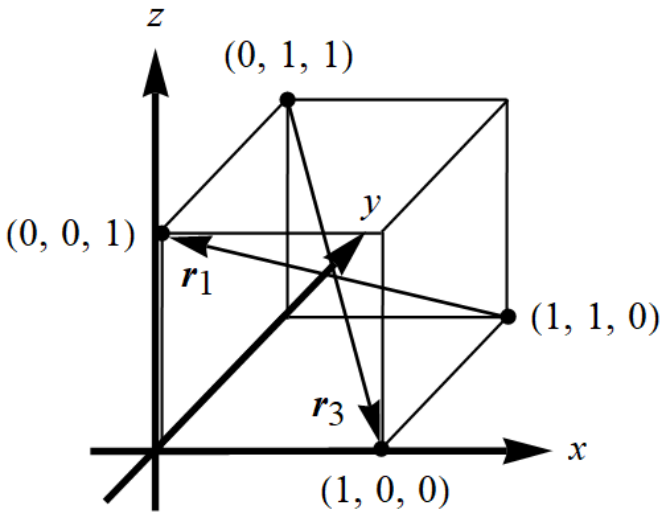
Solve for θ, the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$.

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{|\mathbf{r}_1||\mathbf{r}_2|} \\
&= \frac{\langle -1,-1,1\rangle \cdot \langle -1,1,1\rangle}{\sqrt{(-1)^2+(-1)^2+1^2}\sqrt{(-1)^2+1^2+1^2}} \\
&= \frac{(-1)(-1)+(-1)(1)+(1)(1)}{\sqrt{3}\sqrt{3}} \\
&= \frac{1}{3}
\end{aligned}$$

Therefore, the angle between adjacent body diagonals is

$$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5°.$$

Let $\mathbf{r}_1$ and $\mathbf{r}_3$ be displacement vectors for body diagonals that have their starting points on opposite vertices in a unit cube, for example, those shown below.



$$\mathbf{r}_1 = \langle 0,0,1\rangle - \langle 1,1,0\rangle = \langle -1,-1,1\rangle$$
$$\mathbf{r}_3 = \langle 1,0,0\rangle - \langle 0,1,1\rangle = \langle 1,-1,-1\rangle$$

Use the definition of the dot product between $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\mathbf{r}_1 \cdot \mathbf{r}_3 = |\mathbf{r}_1||\mathbf{r}_3| \cos\alpha$$

Solve for α, the angle between $\mathbf{r}_1$ and $\mathbf{r}_3$.

$$\begin{aligned}
\cos\alpha &= \frac{\mathbf{r}_1 \cdot \mathbf{r}_3}{|\mathbf{r}_1||\mathbf{r}_3|} \\
&= \frac{\langle -1,-1,1\rangle \cdot \langle 1,-1,-1\rangle}{\sqrt{(-1)^2+(-1)^2+1^2}\sqrt{1^2+(-1)^2+(-1)^2}} \\
&= \frac{(-1)(1)+(-1)(-1)+(1)(-1)}{\sqrt{3}\sqrt{3}} \\
&= -\frac{1}{3}
\end{aligned}$$

Therefore, the angle between opposite body diagonals is

$$\alpha = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109.5^\circ.$$

Note that $\alpha + \theta = 180^\circ$.